Active–absorbing-state phase transition beyond directed percolation: A class of exactly solvable models

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We introduce and solve a model of hardcore particles on a one-dimensional periodic lattice which undergoes an active–absorbing-state phase transition at finite density. In this model, an occupied site is defined to be *active* if its left neighbor is occupied and the right neighbor is vacant. Particles from such active sites hop stochastically to their right. We show that both the density of active sites and the survival probability vanish as the particle density is decreased below half. The critical exponents and spatial correlations of the model are calculated exactly using the matrix product ansatz. Exact analytical study of several variations of the model reveals that these nonequilibrium phase transitions belong to a new universality class different from the generic active–absorbing-state phase transition, namely, directed percolation.

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One of the most studied models of nonequilibrium phase transition is directed percolation (DP) [1] defined on a d-dimensional lattice, where an infected site stochastically infects its neighbors in one particular direction. Depending on the infection probability p, the infection may eventually survive (when $p>p_c$) or decay into the absorbing state where no site is infected. Nonequilibrium phase transitions in several other systems, such as reaction diffusion systems [2], depining transitions [3], damage spreading [4], synchronization transition [5], sand-pile models [6], and certain probabilistic cellular automata [7], are known to be in the universality class of DP. It has been conjectured [8] that an "active–absorbing phase transition governed by a fluctuating scalar order parameter" generically belongs to the universality class of DP. There are certain exceptions though. Particle-hole symmetry [9], conservation of parity [10], and symmetry between different absorbing states [11] lead to different universalities. Again in sand-pile models [12], coupling of the order parameter to the conserving height fields [13] results in different critical behavior. Also, conserved lattice-gas (CLG) models [14,15] where the activity field is coupled to the conserved density show critical behavior different from DP. This absorbing-state phase transition in the presence of conserved field is not well understood and most studies in this direction are numerical.

In this paper, we provide an exact analytical solution for a model of hardcore particles on a one-dimensional ring which undergoes an active–absorbing phase transition as the density of particles is changed. The model is defined with a dynamics where a particle from an occupied site hops to the right neighboring site if the left one is occupied. This restricted asymmetric exclusion process (RASEP) leads to a transition from an active phase to an absorbing state as the density of the system falls below $\frac{1}{2}$. The critical exponents of the system at the transition point and spatial correlations have been calculated exactly using the matrix product ansatz (MPA) [16,17]. Some variations of the model, where particles may hop to both directions stochastically, or hop to the right (left) only when it is followed by μ or more particles from left (right) could also be solved exactly. These models, which have same exponents at the transition point, form a new universality class of active–absorbing phase transition different from the generic universality class, namely, DP.

The model is defined on a one-dimensional lattice labeled by sites $i=1,2,\ldots,L$ which are either vacant or occupied with at most one particle; corresponding site variables are taken $s_i=0,1$. A periodic boundary condition is imposed so that $s_{i+L}=s_i$. The dynamics of the system can be described as follows. A particle from a randomly chosen site i is transferred to the right only if $s_{i+1}=0$ and $s_{i-1}=1$. This particle conserving dynamics is thus equivalent to a reaction diffusion system

$$110 \rightarrow 101. \tag{1}$$

We define the activity field at site i as $\phi_i=s_{i-1}s_i(1-s_{i+1})$ which takes values 1 or 0 depending on whether the site i is active, i.e., $s_i=1=s_{i-1}$ and $s_{i+1}=0$. The density of active sites

$$\langle \phi_i \rangle = \langle s_{i-1}s_i(1-s_{i+1}) \rangle \tag{2}$$

is denoted by ρ_a in the thermodynamic limit. A configuration is said to be active if there is at least one active site, otherwise it is called absorbing. For a system of $N=\Sigma_i^L s_i$ particles density is $\rho=\frac{N}{L}$. Clearly, there is only one configuration at $\rho=0$ (and at $\rho=1$) which is absorbing. First let us consider the regime $\rho \leq \frac{1}{2}$ where there are both active and absorbing configurations. Total number of absorbing configurations in this regime is $\frac{L}{L-N}C_N^{L-N}$ and the rest are active. In this regime, the system is arrested in one of these absorbing configurations in steady state resulting in $\rho_a=0$. For $\rho>\frac{1}{2}$, however, there is no absorbing configuration which corresponds to an active phase with fluctuating density of active sites. Thus, ρ_a can be taken as the order parameter of this active–absorbing-state phase transition occurring at $\rho_c=\frac{1}{2}$.

One can describe the dynamics of the model alternatively in terms of the bond variables τ_i, which connects the sites s_i and s_{i+1}. Correspondingly, we choose $\tau_i=2s_i+s_{i+1}$ for four possible combinations (s_i,s_{i+1}). Note that every configuration

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$\{s_i\}$ can be uniquely translated to $\{\tau_i\}$ and vice verse. The dynamics can be described in terms of τ_i as a reaction system

$$321 \to 213, \quad 320 \to 212. \tag{3}$$

The fact that τ_i and τ_{i+1} have a common site s_{i+1} puts certain restrictions on the allowed configurations. However, one need not bother about those as any configuration translated from $\{s_i\}$ automatically satisfies these restrictions and the dynamics (3) respects the same.

To get a steady-state distribution for the reaction system, either Eq. (1) or Eq. (3) which has three site dynamics, we generalize the formulation of matrix product ansatz [16,17] which is commonly used for a two site dynamics. This generalization is different from what has been discussed earlier [18]. Let us describe the formulation in generic terms before using it in this specific problem. In MPA, first a configuration $\{\sigma_1, \sigma_2, \ldots \sigma_L\}$ is translated to a product of matrices by replacing each σ_i by a matrix A_{σ_i} and an ansatz is made that for a periodic system, the un-normalized weight in the steady state is given by

$$f(\sigma_1, \sigma_2, \ldots \sigma_L) = \mathrm{Tr}[A_{\sigma_1}, A_{\sigma_2}, \ldots A_{\sigma_L}]. \tag{4}$$

This ansatz could provide an exact solution for any three site dynamics if one can find matrices A_σ such that steady-state weight (4) satisfies the corresponding master equation in steady state,

$$\begin{aligned} 0 &= \frac{d}{dt} f(\sigma_1, \sigma_2 \ldots \sigma_L) \\ &= \sum_{i,\sigma'} W(\sigma_i' \sigma_{i+1} \sigma_{i+2} \to \sigma_i \sigma_{i+1} \sigma_{i+2}) f(\ldots \sigma_i' \sigma_{i+1} \sigma_{i+2} \ldots) \\ &\quad - \sum_{i,\sigma'} W(\sigma_i \sigma_{i+1} \sigma_{i+2} \to \sigma_i' \sigma_{i+1}' \sigma_{i+2}') f(\ldots \sigma_i \sigma_{i+1} \sigma_{i+2} \ldots). \end{aligned}$$

Here Ws are the transition rates for the three site dynamics. Right-hand side of the master equation can be arranged to vanish for any generic three site dynamics if

$$\begin{aligned} &\sum_{\sigma'} W(\sigma_i' \sigma_{i+1} \sigma_{i+2} \to \sigma_i \sigma_{i+1} \tau_{i+2}) f(\ldots \sigma_i' \sigma_{i+1} \sigma_{i+2} \ldots) \\ &\quad - \sum_{\sigma'} W(\sigma_i \sigma_{i+1} \sigma_{i+2} \to \sigma_i' \sigma_{i+1}' \sigma_{i+2}') f(\ldots \sigma_i \sigma_{i+1} \sigma_{i+2} \ldots) \\ &\quad = \mathrm{Tr}[\ldots \tilde{A}_{\sigma_i} \tilde{A}_{\sigma_{i+1}} A_{\sigma i+2} \ldots] - \mathrm{Tr}[\ldots A_{\sigma_i} \tilde{A}_{\sigma_{i+1}} \tilde{A}_{\sigma_{i+2}} \ldots], \end{aligned} \tag{5}$$

where $\tilde{A}_\sigma$ are auxiliary matrices. Equation (5) is a generalization of the cancellation procedure introduced earlier [16] for two site dynamics. Such a cancellation is feasible only when one can find matrices and auxiliaries which satisfy Eq. (5) for a specific dynamics.

Now let us try to apply this generic scheme to the dynamics (1) and (3). In the first case (1), by replacing s_i with a matrix A_{s_i} we find that the cancellation would occur only if

$$A_1 A_1 A_0 = -\tilde{A}_1 \tilde{A}_1 A_0 + A_1 \tilde{A}_1 \tilde{A}_0 = \tilde{A}_1 \tilde{A}_0 A_1 - A_1 \tilde{A}_0 \tilde{A}_1. \tag{6}$$

Note that these algebraic relations cannot be satisfied by nonzero scalars A_0 and A_1, but there are solutions where A_0, A_1 and the auxiliaries are finite-dimensional matrices [19].

Next, for the dynamics (3) with bond variables, we replace τ_i by matrices X_{τ_i} and demand that the generic cancellation scheme (5) should hold for this dynamics (3). Then, the matrices $\{X_\tau\}$ and auxiliaries $\{\tilde{X}_\tau\}$ must satisfy

$$X_3 X_2 X_1 = \tilde{X}_2 \tilde{X}_1 X_3 - X_2 \tilde{X}_1 \tilde{X}_3 = -\tilde{X}_3 \tilde{X}_2 X_1 + X_3 \tilde{X}_2 \tilde{X}_1, \tag{7}$$

$$X_3 X_2 X_0 = \tilde{X}_2 \tilde{X}_1 X_2 - X_2 \tilde{X}_1 \tilde{X}_2 = -\tilde{X}_3 \tilde{X}_2 X_0 + X_3 \tilde{X}_2 \tilde{X}_0. \tag{8}$$

It is not difficult to see that Eqs. (7) and (8) have a scalar solution $X_0=0$, $X_1=X_2=X_3=1$ with auxiliaries $\tilde{X}_0=0$, $\tilde{X}_1=2=\tilde{X}_2$, $\tilde{X}_3=\frac{3}{2}$. Usually the solutions of MPA with one of the matrix being zero are not acceptable as it indicates that certain configurations are never visited in steady state. For example, here $X_0=0$ would mean that steady-state weight is zero for all configurations having two or more consecutive zeros. Thus, to accept solutions with $X_0=0$, we must show *a priori* that the steady state of the above said configurations are *in fact* zero. A direct proof is lengthy. Alternatively, one can prove the same using a mapping of the model to zero range process (ZRP) which is discussed later [below Eq. (21)]. Since it is easy to work with scalars, we choose to continue with dynamics (3). Let us first calculate the partition function keeping in mind that (i) τ_i and τ_{i+1} have a common site s_{i+1} and (ii) $\Sigma_i s_i = N$. The first restriction can be taken care of by defining $X_{\tau_i} = \langle s_i | Y | s_{i+1} \rangle$. Using the above scalar solution, $Y = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then the partition function is

$$Z_{L,N} = {\sum_{\{s_i\}}}' \prod_i^L \langle s_i | Y | s_{i+1} \rangle, \tag{9}$$

where $'$ reminds that the sum is restricted by $\Sigma_i s_i = N$. To evaluate the restricted sum, we go over to grand canonical system which is an ensemble of L site rings, each having a weight z^{2N} where N is the number of particles in the ring takes all possible values. The grand partition function

$$\begin{aligned} \mathcal{Z}_L(z) &= \sum_N^\infty (z^2)^N Z_{LN} = \sum_{\{s_i\}} \prod_i^L \langle s_i | Y | s_{i+1} \rangle z^{s_i + s_{i+1}} \\ &= \mathrm{Tr}(T^L) = \lambda_+^L + \lambda_-^L, \end{aligned} \tag{10}$$

where $T = \left(\begin{smallmatrix} 0 & z \\ z & z^2 \end{smallmatrix}\right)$, and $\lambda_\pm = \frac{z}{2}(z \pm \sqrt{4+z^2})$ are the eigenvalues of T. Average density of particles is then

$$\rho(z) = \lim_{L \to \infty} \frac{\langle N \rangle}{L} = z^2 \frac{d}{d(z^2)} \ln \mathcal{Z} = \lambda_+ / (\lambda_+ + \lambda_-), \tag{11}$$

where $L \to \infty$ limit has been used in the last step. A system with fixed density $\rho = N/L$ would correspond to the choice of z which is consistent with Eq. (11), i.e.,

$$z = \frac{2\rho - 1}{\sqrt{\rho(1-\rho)}}. \tag{12}$$

Now, let us calculate some of the observables. First, the order parameter,

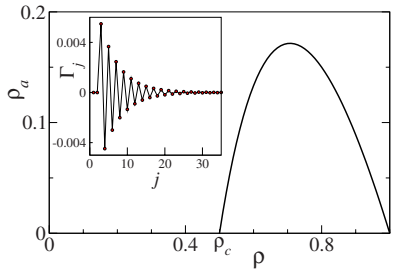

FIG. 1. (Color online) The order parameter ρ_a for RASEP is nonzero for $\rho > 1/2$. Inset shows the decay of correlation function Γ_j defined in Eq. (15).

$$\langle \phi_i \rangle = \frac{1}{\mathcal{Z}_L(z)} \langle 1|T|1\rangle \langle 1|T|0\rangle \langle 0|T^{L-2}|1\rangle = \rho_a \left[\frac{1 - \lambda_+^2 (\lambda_-/\lambda_+)^{L-2}}{1 + (\lambda_-/\lambda_+)^L} \right],$$

where

$$\rho_a = (2\rho - 1)(1 - \rho)/\rho \tag{13}$$

is the order parameter of the system in the thermodynamic limit $L \to \infty$. As $\rho_a = 0$ at $\rho = \rho_c = 1/2$, $\langle \phi_i \rangle$ is independent of L at the critical point. However for $\rho > \rho_c$, $\langle \phi_i \rangle$ has a finite-size correction and it converges to ρ_a exponentially with L.

Any observable can be calculated from the generic $(n+1)$-point correlation function,

$$C_n = \langle s_i s_{i+1} \ldots s_{i+n} \rangle = \lim_{L \to \infty} \frac{1}{\mathcal{Z}_L(z)} \langle 1|T|1\rangle^n \langle 1|T^{L-n}|1\rangle = \rho \left[\frac{2\rho - 1}{\rho} \right]^n. \tag{14}$$

For example, $\langle \phi_i \rangle = \langle s_i s_{i+1}(1 - s_{i+2}) \rangle = C_1 - C_2 = (2\rho - 1)(1 - \rho)/\rho$, which is same as Eq. (13). Now, assuming translational invariance correlation between two active sites separated by j lattice sites, $\Gamma_j = \langle \phi_i \phi_{i+j} \rangle - \langle \phi_i \rangle \langle \phi_{i+j} \rangle$ can be calculated as follows. We have

$$\langle \phi_i \phi_{i+j} \rangle = \frac{1}{\mathcal{Z}_L} \langle 1|T|1\rangle^2 \langle 1|T|0\rangle^2 \langle 0|T^{j-2}|1\rangle \langle 0|T^{L-j-2}|1\rangle = \rho_a^2 \left[1 - \left(\frac{1 - \rho}{\rho} \right)^{j-2} \right],$$

resulting in

$$\Gamma_j(\rho) = -(2\rho - 1)^2 \left[\frac{\rho - 1}{\rho} \right]^j. \tag{15}$$

Note that $\Gamma_j(\rho)$ oscillates with j as shown in the inset of Fig. 1. Such an oscillation is expected as $\phi_i \phi_{i\pm 1} = 0 = \phi_i \phi_{i\pm 2}$ for every i.

Let us calculate the critical exponents of this phase transition. Formally the correlation function is written as $\Gamma(j) \sim \exp(-j/\xi) j^{(-D+2-\eta)}$. Thus, Eq. (14) implies that $\eta = 1$ and that the correlation length $\xi = (\ln \frac{\rho}{1-\rho})^{-1}$ diverges as $\xi \sim (\rho - \rho_c)^{-\nu}$ with $\nu = 1$. From Eq. (13), ρ_a is linear in $(\rho - \rho_c)$ near the critical point. So, the order-parameter exponent $\beta = 1$.

TABLE I. Critical exponents of DP, CDP, and RASEP.

	β	η	ν	β'
DP[a]	0.276486	1.504144	1.0968	0.276486
CDP[a]	0	1	1	1
RASEP	1	1	1	0

[a]Reference [1].

Again, the survival probability $\mathcal{P}$ that a single active site survives in $t \to \infty$ limit vanishes as $\mathcal{P} = (\rho - \rho_c)^{\beta'}$. In RASEP, the activity certainly survives for any density $\rho > \rho_c$; thus, $\beta' = 0$.

For comparison, we have listed all these critical exponents of the model along with those for other known universality classes of active–absorbing phase transitions in Table I. The most well-known and generic universality class of active–absorbing phase transition having a fluctuating scalar order parameter is DP [8]. Models where order parameters obey special conservation laws could differ from DP. One such example is compact-directed percolation (CDP) [9] where the activity field satisfies the particle-hole symmetry. In RASEP, the order parameter which is scalar and fluctuating does not satisfy any special conservation law. That, it shows an active–absorbing phase transition different from DP is surprising. Coupling of this fluctuating order parameter to a conserved field, namely, density, could be a possible cause. In fact, it is well known [12] that in sand-pile models of self-organized critically, the activity field (which is scalar and fluctuating) is coupled to the conserved height field resulting in universality classes different from DP.

To know if this new universality class is stable against perturbations, we have studied several variations of the model by introducing stochasticity both in the direction and rate of particle transfer,

$$110 \to 101, \quad 011 \xrightarrow{p} 101. \tag{16}$$

Naturally, here a site is called active when $s_i = 1$ and either of $s_{i\pm 1} = 0$. Note that CLG in one dimension is a special case of Eq. (16) with $p = 1$ where both forward and backward hopping of particles are allowed. Since this symmetric dynamics satisfies detailed balance [15], all the allowed configurations have the same weight in the steady state. For generic $p \neq 1$, however, the dynamics (16) does not satisfy detailed balance. To obtain the exact steady-state distribution for arbitrary $0 \leq p \leq 1$, we use MPA for three site dynamics described in this paper. Here again, configurations with two or more consecutive zeros are never visited in the steady state, and all other configurations are equally probable. The transition occurs at the critical density $\rho_c = \frac{1}{2}$. The density of active sites in the active phase $(\rho > \rho_c)$ is given by $\rho_a = 2(1 - \rho)(2\rho - 1)/\rho$, which vanishes linearly as $\rho \to \rho_c$ resulting in $\beta = 1$. Other critical exponents $\nu = \eta = 1$ are found to be the same as that of RASEP.

It is worth mentioning that a transformation $1 \leftrightarrow 0$ of Eq. (16) which leads to a dynamics

$$001 \to 010, \quad 100 \overset{p}{\to} 010 \tag{17}$$

also shows a transition at $\rho_c = 1/2$, with order parameter $\rho_a = 2\rho(1-2\rho)/(1-\rho)$ for $\rho < 1/2$. The critical behavior here, as expected, is same as that of RASEP. An interesting variation is when both the dynamics (16) and (17) are present. In this case, we have only two absorbing states {101 010…} and {010 101…} which are symmetric. This may lead to different critical behavior [11], as supported by the numerical studies of these models for $p=1$ [20].

In another variation, a particle from an occupied site hops to its right only when it is followed by μ occupied sites from its left. Thus, Eq. (1) is a special case with $\mu=1$. For finite $\mu=2,3\ldots$ the dynamics are

$$\mu = 2{:}1110 \to 1101,$$

$$\mu = 3{:}11110 \to 11101 \ldots . \tag{18}$$

To use MPA for these dynamics, we have extended our formulation for $(\mu+2)$-site dynamics. The exact results there show that this class of models with $\mu \geq 2$ undergo an active–absorbing phase transition which belong to the same universality class as the system with $\mu=1$. Details of these calculations will be published elsewhere. Here we show a mapping of these models to the ZRP [21] which simplify calculations of some of the observables, such as ρ_a and its distribution.

The ZRP is defined on a periodic one-dimensional lattice with the following dynamics; a single particle from a randomly chosen site (or box) hops to one of its neighbor, say the right one, with rate $u(n)$ which depends on the number of particles n in departure box. To map Eq. (18) to ZRP, we define the vacant sites as boxes, and the number of uninterrupted sequence of 1's to the left of a vacant site as the number of particles in that box. Thus there are N particles which are distributed among $M=L-N$ boxes. Now, dynamics (18) just transfers a particle from a box to its right if the departure box has more than μ particles. Thus,

$$u(n) = \theta(n-\mu), \tag{19}$$

where $\theta(x)$ is the Heaviside theta function. Clearly, when the particles per box $\varrho = \frac{N}{M} < \mu$, the system has at least one configuration where every box contains $\leq \mu$ particles. Such configurations are absorbing and the system is arrested in one of them in the steady state. Thus the critical density is $\varrho_c = \mu$, which corresponds to $\rho_c = \frac{\varrho_c}{1+\varrho_c} = \frac{\mu}{1+\mu}$.

Notice that the steady-state weight in the active phase $\rho > \rho_c$ has product measure,

$$P(n_1, n_2 \ldots n_M) \sim f(n_1) f(n_2) \ldots f(n_M), \tag{20}$$

where n_i is the number of particles in box i and function f is to be determined such that Eq. (20) satisfies the master equation in steady state [21]. In this case, the rate of transfer is independent of number of particles resulting in

$$f(n) = \theta(n+1-\mu). \tag{21}$$

Thus, in steady state, all configurations with *every box* containing μ or more particles are visited with equal probability and all other configurations, which have at least one box containing less than μ particles, are never visited. In particular for RASEP (1), configurations having two or more consecutive zeros are not allowed in steady state (as claimed earlier).

Partition function of the system, in this case, is just the total number of configurations where N particles are distributed in $L-N$ boxes such that each contains at least μ particles

$$Z_{L,N} = C^{N-(\mu-1)(L-N)-1}_{L-N-1}, \tag{22}$$

Note that every configuration $\{s_i\}$ has L translationally equivalent configurations, whereas in ZRP it has only $L-N$ equivalent ones. This raises a multiplicative factor $L/(L-N)$ to the steady-state weight of *every* configuration. We have ignored this factor in Eq. (22) and in further calculations as it does not affect the observables.

This mapping allows the calculation of fluctuations in the number of active sites $N_a = \Sigma_{i=1}^{L} \phi_i = \Sigma_{i=1}^{M} \theta(n_i - \mu)$. In a system of N particles distributed among M boxes probability of finding N_a boxes which have more than μ particles with a restriction that every box contains at least μ particles is given by a hypergeometric distribution,

$$P(N_a) = \frac{1}{Z_{L,N}} C^{N-\mu(L-N)-1}_{N_a-1} C^{L-N}_{N_a}. \tag{23}$$

Mean and variance of this distribution are related to the order parameter and its fluctuation respectively as

$$\rho_a = \lim_{L\to\infty} \frac{\langle N_a \rangle}{L} = \frac{[\rho - \mu(1-\rho)](1-\rho)}{\rho - (\mu-1)(1-\rho)}, \tag{24}$$

$$\Delta\rho_a = \lim_{L\to\infty} \frac{1}{L}(\langle N_a^2 \rangle - \langle N_a \rangle^2) = \frac{\rho_a^2}{\rho - (\mu-1)(1-\rho)}. \tag{25}$$

Equation (24) provides an exact expression of ρ_a for the generic model, which vanishes linearly as ρ approaches $\rho_c = \frac{\mu}{1+\mu}$. From Eq. (25) it is clear that the fluctuation vanishes quadratically as $\rho \to \rho_c$. Contrary to other known continuous transitions, here the transitions are not associated with diverging fluctuation of the order parameter but it is associated with a diverging correlation length.

Of course, one can extend these models to incorporate particle transfer to both directions. A model of ZRP having a generic threshold μ and unbiased particle transfer has been studied earlier [22] in d dimensions. Characteristic critical exponets, in the context of sand-pile models, have been discussed.

In conclusion, we have introduced a class of models in one dimension where a particle can move to a vacant neighboring site in one direction only if it is followed by μ number of particles in the other direction. We extended the matrix product ansatz to generic three site dynamics and apply the formalism to the simplest version of the model with $\mu=1$ to get the exact steady-state distribution. We show that these models undergo a continuous active–absorbing phase transition when density of particles is decreased below $\rho_c = \mu/(1+\mu)$. Interestingly, the fluctuation of the order parameter here does not show any divergence at the transition point, whereas

active sites are found to be correlated within a length scale ξ which diverges as critical density is approached from above. Critical exponents of this active–absorbing phase transition, which are calculated analytically, are found to be different from the generic universality class, namely, directed percolation. We argue that the fluctuating scalar order parameters in these models are coupled to the density field which is conserved, which could be a possible reason why these models differ from the DP universality class.

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